Continuous measurements on continuous variable quantum systems: The Gaussian description

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The Gaussian state description of continuous variables is adapted to describe the quantum interaction between macroscopic atomic samples and continuous-wave light beams. The formalism is very efficient: a non-linear differential equation for the covariance matrix of the atomic system explicitly accounts for both the unitary evolution, the dissipation and noise due to the atom-light interaction, and the back-action due to homodyne optical detection on the beam after its interaction with the atoms. Applications to atomic spin squeezing and estimation of unknown classical parameters are presented, and extensions beyond the Gaussian states are discussed.

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I. INTRODUCTION

Pulses of light, large atomic ensembles, and collections of more than, say, hundred trapped ions, are quantum systems where the behavior of various collective degrees of freedom is well described by quantities which have continuous spectra, i.e., the systems may be described by collective effective position and momentum variables. The demonstration of quantum control of these systems varies from studies of squeezing and entanglement, over storage and retrieval of optical information in gases to high precision probing of classical properties in atomic magnetometry, atomic clocks and inertial sensors. Control is exercised via tunable interactions, by state reduction due to measurements on the systems, and by feed-back schemes in connection with measurements.

In quantum optics, the quantum properties of a continuous beam of light are normally described in the Heisenberg picture, where field operators are expressed (often in the frequency domain) in terms of incoming vacuum fields with standard correlation functions. This input–output formalism leads, e.g., to the noise spectrum of a squeezed light beam[1]. This approach accounts for the results one obtains if measurements are carried out directly on the beam, but it has been technically very difficult to describe the situation where the light beam is made subject to interaction with another quantum system and is subsequently measured. The measurement record is stochastic, and a real-time description of the measurement back-action on the probed quantum system is normally referred to quantum trajectory or Monte Carlo wave function treatments in the Schrödinger picture, which are incompatible with the frequency domain Heisenberg representation of the optical beam.

In Fig. 1, we display the interaction between light and atoms. A Gaussian state analysis was introduced recently to deal, in general terms, with the quantum properties of these systems, and tools were developed to handle interactions and measurements which preserve the Gaussian state character[2, 3]. As we shall illustrate below, the Gaussian description is useful because (i) it handles the interaction between atoms and a quantized continuous-wave (cw) beam of light and (ii) it allows a description of measurement induced back–action in real time. This description thus provides a useful approach to a long standing problem in quantum optics, and it presents a theoretical treatment of physical systems and interactions of high current interest. The description is restricted to Gaussian states. A cw laser beam described by a coherent state and squeezed and quantum correlated optical beams created by down conversion are Gaussian in the field canonical variables and hence readily incorporated in our treatment. Turning now to the atoms, our approach does not describe the interaction with a single ion or atom, but a very accurate mapping exists between macroscopically spin-polarized atomic samples and a single harmonic oscillator.

In this work, we describe the practical application of the Gaussian state formalism to continuous variable systems, allowing full account of back-action due to measurement, noise, losses and inhomogeneities of the systems. The formalism is illustrated by a discussion of explicit examples concerning spin squeezing, magnetometry and entanglement. A whole tool-box can be created, describing the effect of frequency filters, finite band-width sources and detectors, finite efficiency detection, and dark counts, simply by adding extra reservoir modes. In practice, the Gaussian state for a system of n quantum harmonic oscillators, representing a number of optical beams and atomic components, is described by $2n$ mean values for the quadrature components and by a $2n \times 2n$ covariance matrix. While the evolution during measurements of mean values is stochastic, the covariance matrix is propagated in time in a deterministic way (see Sec. 2). It is a remarkable advantage of the Gaussian state description that extra physical systems and reservoir modes can be included at only little expense (two extra rows and columns in the covariance matrix per mode). In the last paragraphs of this work, it will be discussed how to develop a theory for continuous variable systems where the

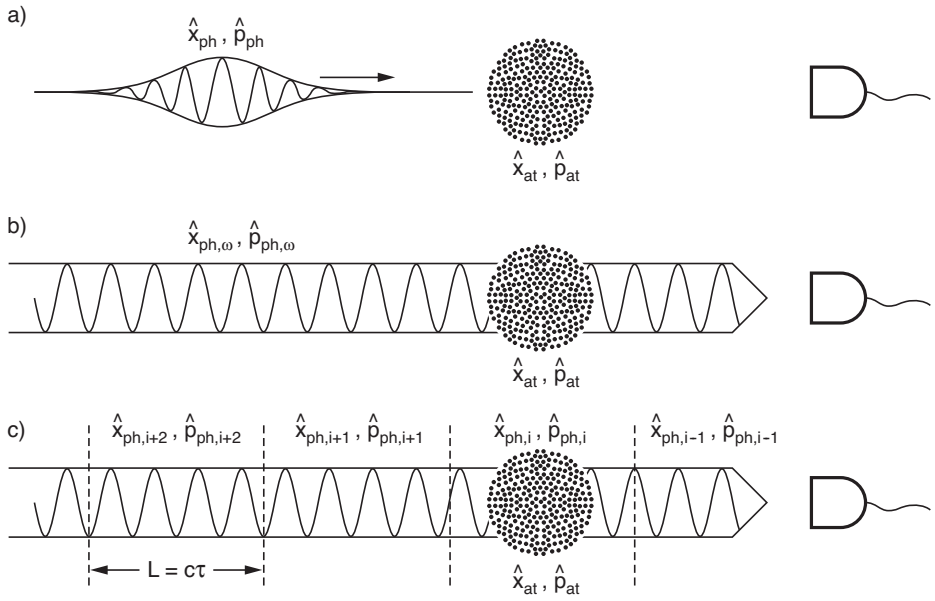

FIG. 1: Atom-light interaction. In the figure, we display a cloud of atoms described by collective continuous variables $\hat{x}_{\rm at}, \hat{p}_{\rm at}$, interacting with (a) a pulse of light and (b)-(c) a continuous wave of light. In (a) the assumption of an accurate description of the light pulse in terms of just a single mode $(\hat{x}_{\rm ph}, \hat{p}_{\rm ph})$ leads to a simple and natural description in the time domain of both the interaction of the pulse with the atoms as well as the detection process. In quantum optics, the continuous beam of light in (b) is normally described in the frequency domain, say, by canonical operators $\hat{x}_{{\rm ph},\omega}, \hat{p}_{{\rm ph},\omega}$. The interaction with the atoms and the measurement process, on the other hand, is more readily described in the time domain, and as discussed in detail in the text, it is technically difficult to pass from the frequency domain for the light operators to the time domain for the description of interaction and measurement process. To circumvent this problem, we introduce an effective description of the integral system in the time domain as indicated in (c). Here the beam is divided into segments of duration τ and length $L = c\tau$ each of which is assumed to be short enough to be accurately described by a single mode $\hat{x}_{{\rm ph},i}, \hat{p}_{{\rm ph},i}$, and the interaction with the atoms and the measurement is described by a succession of interactions with the individual beam segments.

Gaussian description breaks down, either because of the interactions involved, because of the measurement schemes, or because of coupling of a small discrete system to collective continuous degrees of freedom.

II. TIME EVOLUTION OF GAUSSIAN STATES, GENERAL THEORY

In this section, we introduce the Gaussian description in a general setting using existing results[2, 3, 4]. Sections 2.1 and 2.2 deal with the evolution of continuous variable systems due to a bilinear Hamiltonian and linear losses. This evolution can be solved by an affine transformation in time of the canonical operators, and all system properties are given by their mean values and their covariance matrix for which an exact treatment is provided. Section 2.3 deals with the effect of measurements on the system. The update of the system state vector or density operator conditioned on a measurement outcome is non-trivial in the most general case, but as we shall see, Gaussian states transform into other Gaussian states in a well described manner under homodyne detection on part of the system, and in this case the mean values and the covariance matrix still provide all properties of the system.

A. Time evolution due to a bilinear Hamiltonian

Let $\hat{\boldsymbol{y}} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \ldots, \hat{x}_n, \hat{p}_n)^T$ denote the column vector of $2n$ variables with canonical commutators $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$, and let $\hat{H} = \hat{H}(\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \ldots, \hat{x}_n, \hat{p}_n)$ denote the Hamiltonian of the system. We shall assume that $\hat{H}$ is bilinear in the canonical variables. Heisenberg's equations of motion during time τ are then solved by a linear transformation of the operators by the matrix $\mathbf{S}_\tau$

$$\hat{\boldsymbol{y}}(t+\tau) = \mathbf{S}_\tau \hat{\boldsymbol{y}}(t). \tag{1}$$

The same transformation applies to the vector of mean values $\mathbf{m} \equiv \langle \hat{\boldsymbol{y}} \rangle$, $\mathbf{m}(t+\tau) = \mathbf{S}_\tau \mathbf{m}(t)$. From Eq. (1) and the definition of the covariance matrix $\gamma_{ij} \equiv 2\mathrm{Re}\,\langle (\hat{y}_i - \langle \hat{y}_i \rangle)(\hat{y}_j - \langle \hat{y}_j \rangle)\rangle$, we directly verify that $\boldsymbol{\gamma}$ transforms as

$$\boldsymbol{\gamma}(t+\tau) = \mathbf{S}_\tau \boldsymbol{\gamma}(t) \mathbf{S}_\tau^T \tag{2}$$

under the interaction.

B. Time evolution due to dissipation and noise

In the absence of dissipation Eq. (2) determines the evolution of the covariance matrix. In realistic situations, however, there will be sources of dissipation and noise. Dissipation leads to a reduction in the mean values of the canonical variables, and as is known from the quantum theory of damping and the fluctuation-dissipation theorem of statistical mechanics, such a reduction must be accompanied by fluctuations. In the quantum domain we must, e.g., fulfill the Heisenberg uncertainty relations, also when the mean values are reduced. The generalization of Eq. (2) to the noisy case reads for small τ

$$\boldsymbol{\gamma}(t+\tau) = \mathbf{L}_\tau \mathbf{S}_\tau \boldsymbol{\gamma}(t) \mathbf{S}_\tau^T \mathbf{L}_\tau + \mathbf{N}_\tau, \tag{3}$$

where $\mathbf{L}_\tau$ describes the reduction of the mean values, $\mathbf{m}(t+\tau) = \mathbf{L}_\tau \mathbf{S}_\tau \mathbf{m}(t)$, and where $\mathbf{N}_\tau$ is the associated noise. In examples below, we shall give explicit forms of these matrices.

If the state of the system is initially a Gaussian state, i.e., its Wigner function for the canonical variables is a Gaussian function, the evolution due to a bilinear Hamiltonian preserves the Gaussian character. The same is true for linear damping of an optical field mode, and as validated by a calculation and more detailed discussion[5] it also holds to an excellent approximation for atomic decay models.

C. Time evolution due to a homodyne measurement event

The above arguments were based on the Heisenberg picture evolution of the canonical operators, but the evolution due to measurements is more conveniently described as state reduction in a Schrödinger picture representation of the system state vector or density operator. A general representation of the state, pure or mixed, of a collection of harmonic oscillators is provided by the Wigner function $\mathcal{W}(\xi)$ with $\xi \equiv (\xi_1, ...\xi_{2n}) \in \mathbb{R}^{2n}$. This function is connected with the density matrix in position or momentum representations by a Fourier-transformation, and it provides a good intuitive picture of the phase space distribution of the system. In fact, the expectation value of any symmetrically ordered function $F_{\mathrm{sym}}(\hat{x}_1, \hat{p}_1, ...\hat{x}_n, \hat{p}_n)$ (F_{sym} is the average of all the ways of ordering the operators defining $F(\hat{x}_1, \hat{p}_1, ...\hat{x}_n, \hat{p}_n)$), is given by the pseudo-classical expression:

$$\langle F_{\mathrm{sym}}(\hat{x}_1, \hat{p}_1, ...\hat{x}_n, \hat{p}_n)\rangle = \int d^{2n}\xi \; \mathcal{W}(\xi) F(\xi). \tag{4}$$

We recall that we aim at a description of the state of an atomic sample subject to interaction with an optical beam which is being probed after the interaction. We hence address what happens to the quantum state of the remaining system when one of the sub-systems (with a conjugate pair of observables $\hat{x}_n, \hat{p}_n$) is subject to a measurement. Examples of measurements are positive operator valued measures with coherent state outcomes, homodyne detection which projects the measured sub-system onto a position or momentum eigenstate (equivalent to the limit of a strongly quadrature squeezed state), and number state detection. Such measurements project the $(\hat{x}_n, \hat{p}_n)$ sub-system onto a particular state which we can also describe by a Wigner function $\mathcal{W}_{\mathrm{meas}}(\xi_{2n-1}, \xi_{2n})$. The state of the remaining system conditioned on the outcome leading to this particular state is

$$\mathcal{W}_{\mathrm{cond}}(\xi_1, \ldots, \xi_{2n-2}) = \frac{\int d\xi_{2n-1} d\xi_{2n}, \mathcal{W}(\xi_1, \ldots, \xi_{2n}) \mathcal{W}_{\mathrm{meas}}(\xi_{2n-1}, \xi_{2n})}{P_{\mathrm{meas}}}, \tag{5}$$

with $P_{\mathrm{meas}} = \int d\xi_1 \ldots d\xi_{2n} \mathcal{W}(\xi_1, \ldots, \xi_{2n}) \mathcal{W}_{\mathrm{meas}}(\xi_{2n-1}, \xi_{2n})$.

Now turning to the Gaussian states, a series of simplifications occur. For example, the Wigner functions $\mathcal{W}_{\mathrm{meas}}(\xi_{2n-1}, \xi_{2n})$ for coherent and squeezed states are Gaussian functions of the variables. This implies, that if the initial Wigner function is a Gaussian function of the variables, this property is maintained by the homodyne detection process. Generally, the Wigner function for a Gaussian state is fully parameterized by the mean values $\mathbf{m}$ and the covariance matrix γ:

$$\mathcal{W}_{\mathrm{Gauss}}(\xi) = \frac{1}{\pi^n} \frac{1}{\sqrt{\det \gamma}} \exp\left[-(\xi - \mathbf{m})^T \gamma^{-1} (\xi - \mathbf{m})\right]. \tag{6}$$

As the Gaussian character is also maintained by the bilinear Hamiltonian and the linear decay processes, we conclude that to describe the time evolution of a system which starts in a Gaussian state, it suffices to provide the time dependent $\mathbf{m}$ and γ.

Since part of the system is being measured upon, and hence disappears from our quantum state, cf. Eq.(5), it makes sense to write the covariance matrix in the form

$$\boldsymbol{\gamma} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix}, \tag{7}$$

where the $(2n-2)\times(2n-2)$ sub-matrix $\mathbf{A}$ is the covariance matrix for the variables $\hat{\boldsymbol{y}}_1 = (\hat{x}_i, \hat{p}_1, \ldots, \hat{x}_{n-1}, \hat{p}_{n-1})^T$ which are not subject to measurement, $\mathbf{B}$ is the 2×2 covariance matrix for the sub-system subject to measurement $\hat{\boldsymbol{y}}_2 = (\hat{x}_n, \hat{p}_n)^T$, and $\mathbf{C}$ is the $2\times(2n-2)$ correlation matrix between the elements of $\hat{\boldsymbol{y}}_1$ and $\hat{\boldsymbol{y}}_2$. According to the above expressions (5)-(6), a measurement of $\hat{x}_n$ transforms $\mathbf{A}$ as[2, 3, 4]

$$\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} - \mathbf{C}(\pi\mathbf{B}\pi)^{-}\mathbf{C}^T, \tag{8}$$

where $\pi = \mathrm{diag}(1,0)$, and where $(\ )^-$ denotes the Moore-Penrose pseudoinverse: $(\pi\mathbf{B}\pi)^- = \mathrm{diag}(B(1,1)^{-1}, 0)$. If we associate with the precise measurement of $\hat{x}_n$ an infinite variance of $\hat{p}_n$ and hence a total loss of correlations between $\hat{p}_n$ and the other observables, this result is equivalent with the Bayesian update of a classical Gaussian probability distribution[6]. We recognize the Moore-Penrose pseudoinverse as the normal inverse of the corresponding covariance matrix, $(\pi\mathbf{B}\pi)^- = \mathrm{diag}(B(1,1), \infty)^{-1}$.

Unlike the covariance matrix update, which is independent of the value measured, the vector $\mathbf{m} = \langle\hat{\boldsymbol{y}}\rangle$ of expectation values will change in a stochastic manner depending on the actual outcome of the measurement. The outcome of the measurement of $\hat{x}_n$ is random, and the measurement changes the expectation value of all other observables due to the correlations represented by the covariance matrix. Let χ denote the difference between the measurement outcome and the expectation value of $\hat{x}_n$, i.e., a Gaussian random variable with mean value zero and variance given by half of the appropriate covariance matrix element $B(1,1)$. It follows again from Eqs. (5)-(6), (and from the corresponding classical theory of multi-variate Gaussian distributions,) that the change of $\mathbf{m}_1 = \langle\boldsymbol{y}_1\rangle$ due to the measurement is given by:

$$\mathbf{m}_1 \mapsto \mathbf{m}_1' = \mathbf{m}_1 + \mathbf{C}_\gamma(\pi\mathbf{B}\pi)^{-}(\chi, \cdot)^T, \tag{9}$$

where we use that $(\pi B\pi)^- = \mathrm{diag}(B(1,1)^{-1}, 0)$, and hence the second entrance in the vector $(\chi, \cdot)$ need not be specified.

D. Time evolution due to continuous homodyne measurements

In the continuous interaction between a cw light beam and a cloud of atoms one faces a situation where a single system (the atoms) is continuously indirectly monitored, e.g., by a homodyne detection of the light field. This raises the problems, mentioned in the introduction, of dealing simultaneously with a continuous beam and measurement induced back-action (see also Fig. 1). We have recently solved this problem for Gaussian states[7] by quantizing the light beam in short segments of duration τ and corresponding length $L = c\tau$. These beam segments are chosen so short that the field in a single segment can be treated as a single mode and such that the state of the atoms interacting with the field does not change appreciably during time τ. The evolution of the atomic system with the entire beam of light is obtained by sequential interaction with subsequent light segments. The generic multi-mode character of the cw beam of light is treated in the Schrödinger picture in time domain rather than in the Heisenberg picture in frequency domain (cf. Fig. 1(c)).

The simplest example of continuous light-atom interaction is the one of a coherent monochromatic beam of light, corresponding to a product state of coherent states in each segment along the beam axis. In this case, the problem simplifies significantly because all segments are in the same trivial state prior to the interaction with the atoms. The segments need not be included formally in the update of the covariance matrix until it is their turn to interact with the atoms. Segments which have already interacted with the atoms may be detected instantly after the interaction, and in practice they are if the detector is placed within meters from the interaction volume. The detected segments then disappear from the formal description of the system. Prior to the interaction with the beam, we thus consider only the atomic covariance matrix $\mathbf{A}$, and in the absence of any correlation with the incident beam segment, the block-off-diagonal matrices in Eq.(7) vanish

$$\mathbf{C} = \mathbb{O}_{2\times(2n-2)} \tag{10}$$

while the field state of the incident segment is characterized by the normal noise properties of the coherent state

$$\mathbf{B} = \mathbb{I}_{2\times 2}. \tag{11}$$

The full covariance matrix is now propagated according to Eq. (3), and the matrix changes to describe the state of the atoms and the optical segment after interaction. To describe the effect on the atoms of the measurement on the field segment, we apply the measurement update formula (8) for the atomic part, and since the field segment has been observed and reduced to classical information, we are ready to turn to the interaction with the next light segment, which conveniently fits into the covariance matrix (7) in the same locations as the previous segment according to Eqs. (10)-(11). This evolution is repeated to describe in real time the interaction with a beam for any extended period of time, and the expectation value and our uncertainty about any variable of the system at the end of the interaction is readily found from the appropriate entrances in the vector $\mathbf{m}$ and the matrix γ.

In the limit of small τ the changes in γ and $\mathbf{m}$ expressed by the update formulae (3), and (8)-(9), are infinitesimally small. In this, suitably defined, continuous limit, the update formulae translate into differential equations. After application of Eq. (3), the sub-matrix $\mathbf{C}$ depends linearly on the elements of $\mathbf{A}$ and as shown in Eq. (21) and the ensuing discussion below, its elements are proportional to $\sqrt{\tau}$. $\mathbf{B}$ is essentially unchanged for short τ, and $\mathbf{A}$ changes linearly with τ. In the limit of infinitesimally small time increments, the update formula may therefore be written as a closed non-linear equation of motion for $\mathbf{A}$:

$$\dot{\mathbf{A}} = \lim_{\tau\to 0^+} \frac{\mathbf{A}' - \mathbf{A}}{\tau} \equiv \mathbf{G} - \mathbf{DA} - \mathbf{AE} - \mathbf{AFA}, \tag{12}$$

with suitably defined matrices $\mathbf{G}, \mathbf{D}, \mathbf{E}, \mathbf{F}$. This equation is an example of a so-called matrix Ricatti equation[8], and by the decomposition $\mathbf{A} = \mathbf{WU}^{-1}$, it can be rewritten in terms of two coupled linear equations $\dot{\mathbf{W}} = -\mathbf{DW} + \mathbf{GU}$, and $\dot{\mathbf{U}} = \mathbf{FW} + \mathbf{EU}$. Below, we shall see examples of analytical solutions to the problem based on these equations.

III. APPLICATION OF THE GAUSSIAN FORMALISM TO ATOM-LIGHT INTERACTION

The Gaussian formalism can be applied to describe the interaction between atomic samples and optical beams. In our examples, we consider optical Faraday rotation, which probes the collective spin ground state of a gas of atoms. To introduce the transition to an effective Hamiltonian expressed in terms of canonical variables, we discuss in some detail the interaction of an atomic ensemble with a pulse, or segment, of light.

A. Stokes vector and canonical conjugate variables for light

To make the discussion simple, at first only a single atomic sample and a single pulse or segment of a light beam will be considered. In Faraday rotation experiments, one uses light, which is linearly polarized along the, say, x-axis. The interesting quantum degree of freedom of the light pulse is not the field amplitude itself, but the intensity difference between the linearly polarized components along 45 and 135 degree directions in the xy plane, and between the two circularly polarized components with respect to the z-axis. These components are equally populated on average, but as every single x-polarized photon can be expanded as a superposition of single photon states of either pair of polarizations, their populations will fluctuate according to a binomial distribution. For a pulse with a definite number $N_{\rm ph}$ of photons, one may represent these populations conveniently by the components of the Stokes vector, where the x, y and z-components represent the populations difference of x and y polarizations, 45 and 135 degree polarizations and σ^+ and σ^--polarizations, respectively, i.e.,

$$\hat{S}_x = \frac{\hbar}{2}\left(\hat{a}_x^\dagger \hat{a}_x - \hat{a}_y^\dagger \hat{a}_y\right) = -\frac{\hbar}{2}\left(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+\right), \tag{13}$$

$$\hat{S}_z = \frac{\hbar}{2}\left(\hat{a}_x^\dagger \hat{a}_y + \hat{a}_y^\dagger \hat{a}_x\right) = -\frac{\hbar}{2i}\left(\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+\right), \tag{14}$$

$$\hat{S}_z = \frac{\hbar}{2i}\left(\hat{a}_x^\dagger \hat{a}_y - \hat{a}_y^\dagger \hat{a}_x\right) = \frac{\hbar}{2}\left(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-\right). \tag{15}$$

Since the light is assumed to be linearly polarized along the x axis, $\hat{S}_x$ may be treated classically and from Eq. (13), $\hat{S}_x/\hbar = S_x/\hbar = N_{\rm ph}/2$. The Stokes vector components obey the commutator relations of a fictitious spin, and the variance of the binomial distributions are in precise correspondence with the quantum mechanical uncertainty on $\hat{S}_y$ and $\hat{S}_z$, achieving the Heisenberg limit $\mathrm{Var}(\hat{S}_y)\mathrm{Var}(\hat{S}_z) = |\langle\hbar\hat{S}_x\rangle|^2/4$.

We assume that S_x remains large and essentially unchanged during the interaction with the atomic gas, and we can then introduce the effective position and momentum operators

$$(\hat{x}_{\rm ph}, \hat{p}_{\rm ph}) = \left(\hat{S}_y/\sqrt{|\langle\hbar S_x\rangle|}, \hat{S}_z/\sqrt{|\langle\hbar S_x\rangle|}\right), \tag{16}$$

which fulfill the standard commutator relation $[\hat{x}_{\rm ph}, \hat{p}_{\rm ph}] = i$ and resulting uncertainty relation. These are the canonical conjugate variables that we wish to describe by the formalism outlined in the previous section. The initial binomial distributions of $\hat{S}_y, \hat{S}_z$ approach Gaussian distributions in the limit of large photon numbers. Moreover, the fact that the uncertainty relation is minimized in the initial state implies that this state is a Gaussian state, i.e., the Wigner function for the field is a Gaussian function[9].

B. Atom-light interaction

The physical system of interest consists of one or more macroscopic ensembles of trapped atoms interacting off-resonantly with one or more laser beams. We consider the usual electric dipole interaction between the atoms and the quantized field. First, the off-resonant coupling of the atoms with the light field is expanded in transition operators between the ground ($|FM\rangle$) and excited ($|F'M'\rangle$) hyperfine states (several excited states with different F' may be coupled to the ground state). Then, the atomic coherences pertaining to the excited states are expressed by the light fields and ground state coherences by adiabatic elimination using Heisenberg's equations of motion for the slowly varying operators. This procedure generally allows us to derive a dispersive effective Hamiltonian[10, 11], which for the $N_{\rm at}$ atoms reads[12]

$$\begin{aligned}\hat{H}_{{\rm int},\tau} = \sum_{j=1}^{N_{\rm at}} \sum_{M=-F}^{F} [&\Big(c_{+,M}(\Delta)\hat{a}_+^\dagger\hat{a}_+ + c_{-,M}(\Delta)\hat{a}_-^\dagger\hat{a}_-\Big)|FM\rangle_j\langle FM| \\ &+ b_M(\Delta)\left(\hat{a}_-^\dagger\hat{a}_+|FM+1\rangle_j\langle FM-1| + \hat{a}_+^\dagger\hat{a}_-|FM-1\rangle_j\langle FM+1|\right)],\end{aligned} \tag{17}$$

where field creation and annihilation operators for σ^+ and σ^--polarized photons have been introduced. The first two terms describe the ac Stark shift of the ground state $|FM\rangle$ caused by the coupling to the excited $|F'M\pm1\rangle$ states by the two field components. The coupling coefficients are given by $c_{\pm,M}(\Delta) = -2\hbar\sum_{F'}(g^{\pm}_{FM;F'M'})^2/\Delta_{F'}$ where $\Delta_{F'}$ is the detuning of the laser frequency from the upper level, and where the coupling constants $g^{\pm}_{FM;F'M'}$ are the electric dipole coupling matrix elements, $g^{\pm}_{FM;F'M'} = \sqrt{\omega_0/2\hbar\epsilon_0 Ac\tau}\,d^{\pm}_{FM;F'M'}$. These matrix elements contain the 'electric field per photon' for a plane wave field with transverse area A and length $c\tau$, and they involve the spherical tensor components of the dipole operator $\hat{\boldsymbol{d}} = -e\hat{\boldsymbol{r}}$ of the electron, $d^{\pm}_{FM;F'M'} = \langle FM|\hat{d}_\pm|F'M'\rangle$ and $\hat{d}_+ = -(\hat{d}_x + i\hat{d}_y)/\sqrt{2}$, $\hat{d}_- = (\hat{d}_x - i\hat{d}_y)/\sqrt{2}$. The terms in Eq. (17) proportional to $b_M(\Delta) = -2\hbar\sum_{F'} g^{+}_{FM-1;F'M}g^{-}_{FM+1;F'M}/\Delta_{F'}$ describe $\Delta M = \pm2$ Raman transitions involving absorption and stimulated emission of a pair of photons with different polarization.

1. *Spin* 1/2*-case*

For much of the discussion in the rest of this work, we shall restrict ourselves to the case of atoms with only one ground and one excited level which both have total angular momenta $F = F' = 1/2$. The above Hamiltonian simplifies in this case, and noting further that the dipole matrix elements are related to the total spontaneous decay rate Γ of the upper state, $c_{\pm,\mp1/2} = -3\hbar\Gamma\sigma/(2\tau\Delta A)$, with $\sigma = \lambda^2/(2\pi)$ the resonant photon absorption cross section, Eq.(17) then reduces to

$$\hat{H}_{{\rm int},\tau} = -\sum_j \frac{3\hbar\Gamma\sigma}{2\tau\Delta A}\left(\hat{a}_+^\dagger\hat{a}_+\left|-\frac{1}{2}\right\rangle_j\left\langle-\frac{1}{2}\right| + \hat{a}_-^\dagger\hat{a}_-\left|\frac{1}{2}\right\rangle_j\left\langle\frac{1}{2}\right|\right), \tag{18}$$

where the $F = F' = 1/2$ index has been suppressed.

The atomic ensemble is initially prepared with all $N_{\rm at}$ atoms in a superposition $(|-1/2\rangle + |1/2\rangle)/\sqrt{2}$ of the two ground states with respect to the quantization axis z, i.e., the total state of the atoms is initially given by $\left((|-1/2\rangle + |1/2\rangle)/\sqrt{2}\right)^{N_{\rm at}}$. In this state, the system of two-level atoms is described by a collective spin, $\hat{\boldsymbol{J}} = \frac{\hbar}{2}\sum_j \hat{\boldsymbol{\sigma}}_j$, with a component along the x-direction which attains the macroscopic value $\langle \hat{J}_x\rangle = \hbar N_{\rm at}/2$, and with a component along the z-axis, $\hat{J}_z$, which represents the population difference between the $|\pm 1/2\rangle$ states. Similarly, we may use Eq. (15) and represent the operators of the photon field in terms of the collective Stokes vector operator, $\hat{S}_z$. The Hamiltonian can, hence, be rewritten in terms of the collective spin variables for photons and atoms

$$\hat{H}_{{\rm int},\tau} = -\frac{3\Gamma\sigma}{\tau\Delta A}\hat{S}_z\hat{J}_z, \tag{19}$$

where an overall energy-shift proportional to the number of photons in the pulse segment has been neglected.

As for the photons it is convenient to introduce effective atomic position and momentum coordinates

$$(\hat{x}_{\rm at}, \hat{p}_{\rm at}) = \left(\hat{J}_y/\sqrt{|\langle\hbar\hat{J}_x\rangle|}, \hat{J}_z/\sqrt{|\langle\hbar\hat{J}_x\rangle|}\right), \tag{20}$$

for which the initial state is a minimum uncertainty Gaussian state. The last step of this analysis is then to rewrite Eq. (19) in terms of canonical conjugate variables,

$$\hat{H}_{{\rm int},\tau} = \hbar\kappa_\tau \hat{p}_{\rm at}\hat{p}_{\rm ph}, \tag{21}$$

where

$$\kappa_\tau = -\frac{3\Gamma\sigma}{\tau\Delta A}\sqrt{|\langle\hat{S}_x\rangle|}\sqrt{|\langle\hat{J}_x\rangle|}. \tag{22}$$

The Hamiltonian correlates the atoms and the light fields and is bilinear in the canonical variables. Hence the theoretical formalism of Sec. 2 applies. The coupling constant κ_τ is small for realistic parameters, and a coarse grained description, where the atoms interact with one segment of light after the other, will be perfectly valid even for the macroscopic number of photons $N_{\rm ph}$ in each segment required by our Gaussian treatment. Note that $\langle\hat{S}_x\rangle$ is proportional to the number of photons in the beam segment, i.e., to τ, and it follows that $\hat{H}_{{\rm int},\tau}\tau$ is proportional to $\sqrt{\tau}$ yielding a well-defined differential limit in Eq. (12).

We have emphasized the convenience of using Gaussian states, because their Schrödinger picture representation is very efficient and compact. Now, given that every segment of the optical beam becomes correlated with the atomic sample, as a function of time, the joint state of the atom and field has to be specified by a larger and larger number of mean values and second order moments. If no further interactions take place between the atoms and the light after the interaction, there is no need to keep track of the state of the total system. In practice, either the transmitted light may simply disappear or it may be registered in a detection process. In the former case, the relevant description of the remaining system is obtained by a partial trace over the field state, which produces a new Gaussian state of the atoms, which is simply given by removing the photonic lines and columns of the covariance matrix immediately after the interaction update (3). The measurement of the small Faraday rotation of the linearly polarized probe is done by a measurement of the intensity difference between the 45 and 135 degree polarization components, i.e., by a measurement of the $\hat{S}_y \propto \hat{x}_{ph}$ observable, which is precisely the "homodyne" measurement described in section II D. The atomic state is thus described by the corresponding update formula of Eq. (12).

IV. SPIN SQUEEZING IN THE GAUSSIAN DESCRIPTION

With spin squeezed atomic ensembles, i.e., samples where the variance of one of the angular momentum (spin) components is reduced compared with the coherent state value, one has the possibility to measure certain atomic and/or classical parameters beyond the precision set by the standard quantum noise.

The theory of squeezing of the collective atomic spin variable was dealt with in a series of papers[10, 11, 13], and extended to include investigations of quantum non-demolition feedback schemes[14, 15], and inhomogeneous light-atom coupling[16, 17]. In a series of related works[7, 18, 19, 20], spin-squeezing of continuous variable quantum systems has been investigated in the approximation where the atomic and photonic degrees of freedom are described by a Gaussian state.

We are interested in the case, where the polarization rotation of the light field is registered, i.e., the observable $\hat{x}_{\rm ph}$ is measured. The effect of measuring one of the components in a multi-variable Gaussian state is effectively to

produce a new Gaussian state of the remaining variables as discussed in detail in Sec. II. The column vector of the variables for the gas and the photon field reads $\hat{\boldsymbol{y}} = (\hat{x}_{\rm at}, \hat{p}_{\rm at}, \hat{x}_{\rm ph}, \hat{p}_{\rm ph})^T$ and the S-matrix in Eq. (1) is

$$\mathbf{S}_\tau = \begin{pmatrix} 1 & 0 & 0 & \kappa_\tau \\ 0 & 1 & 0 & 0 \\ 0 & \kappa_\tau & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

A. Dissipation and noise

In the probing process there is a small probability that the excited atomic levels which were adiabatically eliminated from the interaction Hamiltonian of Eq. (21) will be populated. If this happens, the subsequent decay to one of the two $M_z = \pm 1/2$ ground states occurs with the rate $\eta = \Phi \frac{\sigma}{A} \left(\frac{\Gamma^2/4}{\Gamma^2/4+\Delta^2} \right)$, where Φ is the photon flux and where the remaining parameters were defined in Sec. III B 1. The consequence of the decay is a loss of spin polarization since a detection of the fluorescence photons in principle can tell to which ground state the atom decayed. If every atom has a probability $\eta_\tau = \eta\tau$ to decay in time τ with equal probability into the two ground states, the collective mean spin vector is reduced by the corresponding factor $\langle \boldsymbol{J} \rangle \to \langle \boldsymbol{J} \rangle (1 - \eta_\tau)$.

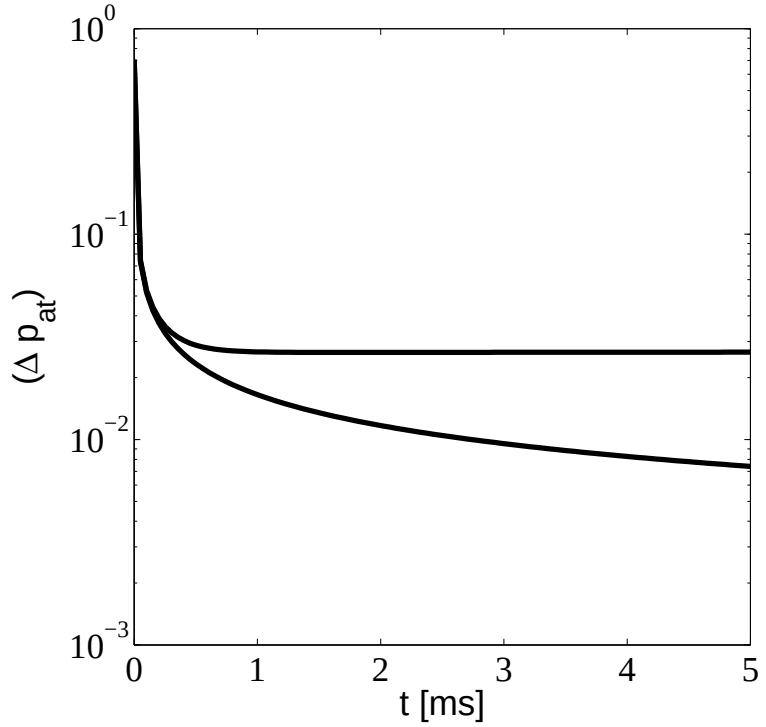

FIG. 2: Uncertainty of $p_{\rm at}$ as function of time during optical Faraday rotation experiment. The effective coupling is $\kappa^2 = 1.83 \times 10^6\,{\rm s}^{-1}$. The lower curve is without inclusion of atomic decay, and the upper curve includes atomic decay with a rate $\eta = 1.7577\,{\rm s}^{-1}$ and photon absorption with a probability $\epsilon = 0.028$. These values correspond, for example, to a 2 mm^2 interaction area, 2×10^{12} atoms, 5×10^{14} photons s^{-1}, 10 GHz detuning, and 852 nm light, appropriate for the ^{133}Cs($6S_{1/2}(F = 4) - 6P_{1/2}(F = 5)$) transition. Factors of order unity related to the coupling matrix elements among different states of the actual Zeeman substructure of Cs are omitted.

Simultaneously, every photon on its way through the atomic gas has a probability for being absorbed[19] $\epsilon = N_{\rm at} \frac{\sigma}{A} \left(\frac{\Gamma^2/4}{\Gamma^2/4+\Delta^2} \right)$ (see Sec. III B 1 for definition of parameters). The effect of these noise contributions were discussed in detail elsewhere[5, 20], and the result for the reduction and noise matrices of the update formula of Eq. (3) reads $\mathbf{L}_\tau = {\rm diag}(\sqrt{1-\eta_\tau}, \sqrt{1-\eta_\tau}, \sqrt{1-\epsilon}, \sqrt{1-\epsilon})$, and $\mathbf{N}_\tau = {\rm diag}(\frac{\hbar N_{\rm at}}{\langle J_x(t) \rangle}\eta_\tau, \frac{\hbar N_{\rm at}}{\langle J_x(t) \rangle}\eta_\tau, \frac{\hbar N_{\rm ph}}{2\langle S_x(t) \rangle}\epsilon, \frac{\hbar N_{\rm ph}}{2\langle S_x(t) \rangle}\epsilon)$ for $\eta_\tau, \epsilon \ll 1$. The factor $\hbar N_{\rm at}/\langle \hat{J}_x(t) \rangle$ initially attains the value 2, and increases by the factor $(1 - \eta_\tau)^{-1}$ in each time step τ. The

factor $\hbar N_{\rm ph}/(2\langle \hat{S}_x(t)\rangle)$ is initially unity, and is approximately constant in time since the light field is continuously renewed by new segments of the light beam interacting with the atoms.

We note that when the classical x-component of the atomic spin is reduced this leads to a reduction with time of the coupling strength $\kappa_\tau \mapsto \kappa_\tau\sqrt{1-\eta_\tau}$ (see Eq. (22)).

B. Solution of Ricatti equation

We now have explicit forms for the matrices needed for our update of the Gaussian states. In the Gaussian description, the problem of spin squeezing may be solved either by the discrete update formulae or analytically from the matrix Ricatti equation. In the latter case, we note that the covariance matrix after n iterations in the noise-less case is

$$\gamma_n = \begin{pmatrix} 2{\rm Var}(\hat{x}_{\rm at}) & 0 & 0 & 0 \\ 0 & 2{\rm Var}(\hat{p}_{\rm at}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{24}$$

We then apply the S-matrix from Eq. (23) and find

$$\mathbf{S}_\tau\gamma_n\mathbf{S}_\tau^\dagger = \begin{pmatrix} 2{\rm Var}(\hat{x}_{\rm at})+\kappa_\tau^2 & 0 & 0 & \kappa_\tau \\ 0 & 2{\rm Var}(\hat{p}_{\rm at}) & 2\kappa_\tau{\rm Var}(\hat{p}_{\rm at}) & 0 \\ 0 & 2\kappa_\tau{\rm Var}(\hat{p}_{\rm at}) & 1+2\kappa_\tau^2{\rm Var}(\hat{p}_{\rm at}) & 0 \\ \kappa_\tau & 0 & 0 & 1 \end{pmatrix}. \tag{25}$$

From this matrix, we determine, to lowest order in τ, $\mathbf{C}(\pi\mathbf{B}\pi)^-\mathbf{C}^T = \kappa_\tau^2 \begin{pmatrix} 0 & 0 \\ 0 & (2{\rm Var}(\hat{p}_{\rm at}))^2 \end{pmatrix}$, insert into Eq. (8), take the continuous limit and use $\kappa^2 = \kappa_\tau^2/\tau$. This procedure leads to the following differential equation for the variance of $\hat{p}_{\rm at}(\propto \hat{J}_z)$: $\frac{d}{dt}{\rm Var}(\hat{p}_{\rm at}) = -2\kappa^2\left({\rm Var}(\hat{p}_{\rm at})\right)^2$, which is readily solved by separating the variables

$${\rm Var}(\hat{p}_{\rm at}) = \frac{1}{2\kappa^2 t + 1/{\rm Var}(\hat{p}_{\rm at,0})}, \tag{26}$$

where ${\rm Var}(\hat{p}_{\rm at,0}) = 1/2$ is the variance of the initial minimum uncertainty state. Note that the solution to the variance of the conjugate atomic variable is ${\rm Var}(\hat{x}_{\rm at}) = \kappa^2 t/2 + {\rm Var}(\hat{x}_{\rm at,0})$ with ${\rm Var}(\hat{x}_{\rm at,0}) = 1/2$. Hence, while $\hat{p}_{\rm at}$ is squeezed, $\hat{x}_{\rm at}$ is antisqueezed to maintain the equal sign in Heisenberg's uncertainty relation.

When dissipation and noise is included the problem may still be solved analytically[20]. The expressions for the variances are quite complicated and will not be given here. Figure 2 shows the spin squeezing as a function of probing time. When atomic decay is not included, the uncertainty in $\hat{p}_{\rm at}$ is a monotonically decreasing function with time. When decay and noise is included, a minimum at $t_{\rm min}$ is reached whereafter the degree of squeezing starts to decrease. On the time scale of the figure, which is chosen to reflect realistic experimental time scales, the increase in ${\rm Var}(\hat{p}_{\rm at})$ is hardly visible.

C. Inhomogeneous coupling

One of the virtues of the Gaussian description of spin squeezing is that it is straightforwardly generalized to handle situations which are hard to approach by standard means. For example, a variation in the intensity of the light beam across the atomic sample and a large photon absorption probability both lead to an inhomogeneous atom-light coupling[17, 20]. To treat such a case, the atomic gas is divided into n slices each with local light-atom coupling strength κ_i. The $2n+2$ dimensional vector of gaussian variables describing the $2n$ collective canonical position and momentum variables for the atoms, and the two collective position and momentum variables for the photon field then reads $\hat{\boldsymbol{y}} = (\hat{x}_{{\rm at},1}, \hat{p}_{{\rm at},1}, \ldots, \hat{x}_{{\rm at},n}, \hat{p}_{{\rm at},n}, \hat{x}_{\rm ph}, \hat{p}_{\rm ph})^T$, and the generalization of Eq. (21) to this case is

$$\hat{H}_{{\rm int},\tau} = \hbar\left(\sum_{i=1}^{n}\kappa_{\tau,i}\hat{p}_{{\rm at},i}\right)\hat{p}_{\rm ph}, \tag{27}$$

where the summation index covers the different slices of atoms. With this Hamiltonian and the atomic decay and photon absorption loss mechanisms, the appropriate $\mathbf{S}_\tau$, $\mathbf{L}_\tau$, and $\mathbf{N}_\tau$ matrices are readily found, and the update formulae of Sec. II (or a slightly modified version thereof for the optically thick gas[20]) may be applied for the determination of the covariance matrix and the mean value vector for the Gaussian variables in $\boldsymbol{y}$. The result of this

calculation is a $2n \times 2n$ atomic covariance matrix, with only minor squeezing in each slice, as the quantum correlations are distributed over the entire sample. One readily obtains the noise properties of the total atomic spin components, but it is more interesting to find the smallest eigenvalue of the covariance matrix, corresponding to a specific spatial mode of the atoms which is maximally squeezed. This mode, indeed, is the one that couples most efficiently to the radiation, and it is hence this smallest eigenvalue that determines the precision with which one can estimate, e.g., the Larmor rotation rate of the collective spin[20].

V. MAGNETOMETRY IN THE GAUSSIAN DESCRIPTION

Precision atomic magnetometry relies on the measurement of the Larmor precession of a spin-polarized atomic sample in a magnetic field[21, 22, 23]. From standard counting statistics arguments, one might expect the uncertainty in such measurements to decrease with the interaction time t and with the number of atoms $N_{\rm at}$ as $1/\sqrt{N_{\rm at}t}$. If, on the other hand, the monitoring of the atomic sample, necessary for the read-out of the estimate of the magnetic field, squeezes the atomic spin, the above limit may be surpassed. In a theoretical analysis[24] it was suggested to estimate a scalar B field by a polarization rotation measurement of a far off-resonant light beam passing through a trapped cloud of spin-1/2 atoms. By quantum trajectory theory[25] combined with the classical theory of Kalman filters[24, 26], the uncertainty in the classical field strength was found[24] to decrease as $1/(N_{\rm at}t^{3/2})$. This proposal was implemented experimentally, and indeed sub-shot-noise sensitivity was found[27]. In our analysis of the experiment[7, 28], we advocated treating all variables, including the magnetic field, as quantum variables, and to assume a Gaussian probability distribution for the classical variable, so that the entire system can be described by the covariance matrix formulation.

In the case of a scalar field directed along the y direction, the effective Hamiltonian of the system is given by

$$\hat{H}_{{\rm int},\tau} = \hbar(\kappa_\tau \hat{p}_{\rm at}\hat{p}_{\rm ph} + \mu_\tau \hat{x}_{\rm at}\hat{B}), \tag{28}$$

where $\mu_\tau = (\tau/\hbar)\beta\sqrt{|\langle \hat{J}_x\rangle|/\hbar}$ is given by the magnetic moment β, and where the B field causes a Larmor rotation of the atomic spin towards the z axis. Figure 3 shows the setup. It is the coupling of the B field to the spin-squeezed

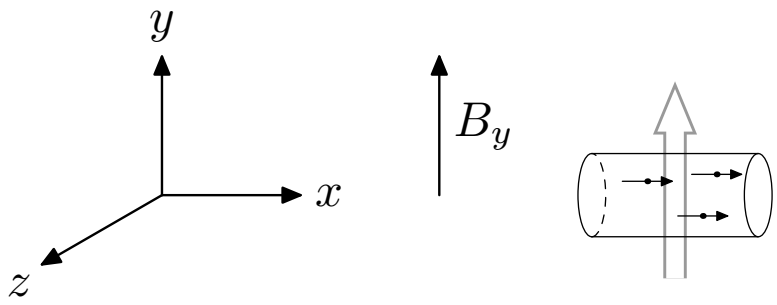

FIG. 3: Setup for measuring the y-coordinate of a magnetic field. This is done by measuring the Farady rotation of a linearly polarized optical beam propagating through the atomic gas.

variable $\hat{p}_{\rm at}$ that makes an improved precision measurement of the magnetic field possible[29].

The vector of variables in the case of a scalar magnetic field is $\hat{\boldsymbol{y}} = (\hat{B}, \hat{x}_{\rm at}, \hat{p}_{\rm at}, \hat{x}_{\rm ph}, \hat{p}_{\rm ph})$, and with the Hamiltonian of Eq. (28), the S-matrix is found to be[7, 28]

$$\mathbf{S}_\tau = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \kappa_\tau \\ -\mu_\tau & 0 & 1 & 0 & 0 \\ 0 & 0 & \kappa_\tau & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{29}$$

As B_y only causes rotation perpendicular to its direction, the variable $\hat{x}_{\rm at} \propto \hat{J}_y$ does not couple to $(B_y, \hat{p}_{\rm at})$ and, hence, we only need to consider a 2×2 system with $\mathbf{y} = (B_y, \hat{p}_{\rm at})^T$. In the noise-less case, the system may now be propagated in time with the discrete update formula of Sec. II. Alternatively we may consider the continuous limit and derive the differential equation for the covariance matrix $\mathbf{A}$ matrix of Eqs. (8)-(12) pertaining to $\mathbf{y} = (B_y, \hat{p}_{\rm at})^T$. The differential equation is on the matrix Ricatti form[8]

$$\dot{\mathbf{A}}(t) = \mathbf{G} - \mathbf{D}\mathbf{A}(t) - \mathbf{A}(t)\mathbf{E} - \mathbf{A}(t)\mathbf{F}\mathbf{A}(t), \tag{30}$$

with $\mathbf{G}=0$, $\mathbf{D}=\left(\begin{smallmatrix}0&0\\ \mu&0\end{smallmatrix}\right)$, $\mathbf{E}=\mathbf{D}^T$, and $\mathbf{F}=\left(\begin{smallmatrix}0&0\\ 0&\kappa^2\end{smallmatrix}\right)$ where $\kappa^2=\kappa_\tau^2/\tau$ and $\mu=\mu_\tau/\tau$. As may be checked by insertion, the solution to Eq. (30) is $\mathbf{A}_\gamma=\mathbf{W}\mathbf{U}^{-1}$, where $\dot{\mathbf{W}}=-\mathbf{D}\mathbf{W}+\mathbf{G}\mathbf{U}$ and $\dot{\mathbf{U}}=\mathbf{F}\mathbf{W}+\mathbf{E}\mathbf{U}$. The resulting solution for the variance of the B field reads:

$$\begin{aligned}\mathrm{Var}(\hat{B}(t)) &= \frac{\mathrm{Var}(\hat{B}_0)(\kappa^2 t+1)}{\frac{1}{6}\kappa^4\mu^2\mathrm{Var}(\hat{B}_0)t^4+\frac{2}{3}\kappa^2\mu^2\mathrm{Var}(\hat{B}_0)t^3+\kappa^2 t+1} \\ &\to_{t\to\infty} \frac{6}{\kappa^2\mu^2 t^3}\propto\frac{1}{N_{\mathrm{at}}^2\Phi t^3},\end{aligned} \tag{31}$$

The presence of noise[28] reduces the asymptotic decrease in the uncertainty with time from $1/t^3$ to $1/t$. Figure 4

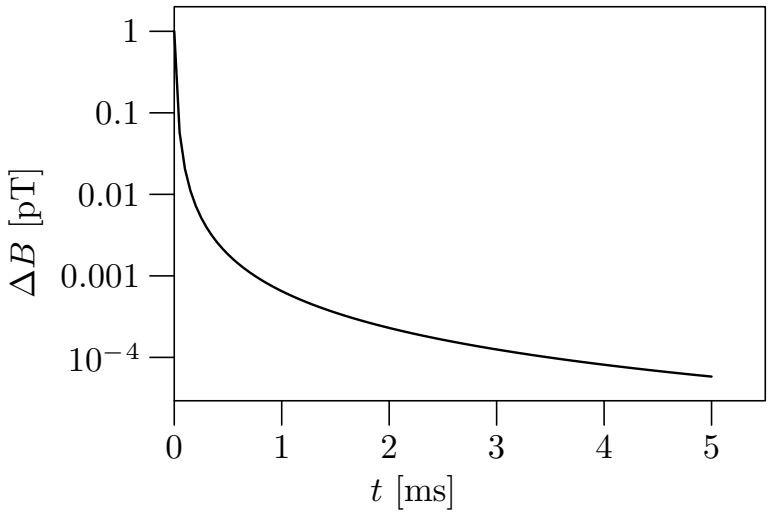

FIG. 4: Uncertainty of B field as a function of time. The value at $t=5\,\mathrm{ms}$ is $\Delta B_y=5.814\times10^{-5}\,\mathrm{pT}$. We have chosen a segment duration $\tau=10^{-8}$ s and corresponding field parameters $\kappa_\tau^2=0.0183$ and $\mu_\tau=8.8\times10^{-4}$.

shows the decrease in the uncertainty of the B field with time in a calculation with physically realizable parameters.

The new concept introduced in estimating the value of the classical B field is to treat the field itself as a quantum variable. Such an approach is not incompatible with the assumption that it is a classical parameter. We may imagine a canonically conjugate variable to B having an uncertainty much larger than required by Heisenberg's uncertainty relation and/or additional physical systems, entangled with the B-variable, in which cases the B-distribution is indeed incoherent and "classical". Also, one may argue that all classical variables are quantum mechanical variables for which a classical description suffices, and hence our theory provides the correct estimator according to the quantum theory of measurements: quantum mechanics dictates that the quantum state provides all the available knowledge about a system, and any estimator providing a tighter bound hence represents additional knowledge equivalent to a local hidden variable, and this is excluded by quantum theory. It is of course crucial that our measurement scheme corresponds to a quantum non-demolition (QND) measurement, i.e., we assume that there is not a free evolution of the B-field induced by its conjugate variable which may thus remain unspecified. It is also this QND property of the measurement scheme that implies a monotonic reduction of the uncertainty of B which is consistent with the classical parameter estimation (we can not unlearn what we have already learnt about B), unlike, e.g., the uncertainty of the atomic $\hat{x}_{\mathrm{at}}$ variable which must increase when $\mathrm{Var}(\hat{p}_{\mathrm{at}})$ is reduced and when the atoms undergo spontaneous decay.

VI. ENTANGLEMENT IN THE GAUSSIAN DESCRIPTION

The theoretical proposal[30, 31] and the subsequent experimental demonstration[32] that by using only coherent light, entanglement can be generated between distant free space atomic ensembles, has attracted much attention in the quantum information community. The primary reason being that entanglement in such macroscopic ensembles of atoms is robust and easy to make because of the effective and tunable atom-light coupling. Theoretical analyses of entanglement with continuous variables has been performed in the Heisenberg picture[30], and the state vector dynamics for a few tens of atoms has been considered by quantum trajectory methods[33, 34]. Also the Gaussian description has been successfully applied to describe the entanglement generation between two macroscopic atomic ensembles due to continuous probing of collective spin variables by optical Faraday rotation[5].

Generally, in the entanglement setup, the two gasses are polarized along opposite directions, say the positive and negative x axis. This means that the classical x components of the collective spin vectors are given by $J_{x,1} = \hbar N_{\mathrm{at},1}/2 \equiv J_x$, and $J_{x,2} = -J_x$, and the Gaussian description is applicable with the following vector of canonical quantum variables $\hat{\boldsymbol{y}} = (\hat{x}_{\mathrm{at},1}, \hat{p}_{\mathrm{at},1}, \hat{x}_{\mathrm{at},2}, \hat{p}_{\mathrm{at},2}, \hat{x}_{\mathrm{ph}}, \hat{p}_{\mathrm{ph}})^T = (\frac{\hat{J}_{y,1}}{\sqrt{\hbar|J_x|}}, \frac{\hat{J}_{z,1}}{\sqrt{\hbar|J_x|}}, -\frac{\hat{J}_{y,2}}{\sqrt{\hbar|J_x|}}, \frac{\hat{J}_{z,2}}{\sqrt{\hbar|J_x|}}, \frac{\hat{S}_y}{\sqrt{\hbar|S_x|}}, \frac{\hat{S}_z}{\sqrt{\hbar|S_x|}})^T$. The Hamiltonian for either sample is given by Eq. (21). To model the light-atom interaction, the light beam is divided into segments as discussed in Sec. II D. The S-matrices $\mathbf{S}_{\tau,1}$ and $\mathbf{S}_{\tau,1}$ for the two gasses are readily found from Heisenberg's equation of motion for the variables in $\boldsymbol{y}$, and combined to $\mathbf{S}_\tau = \mathbf{S}_{\tau,1}\mathbf{S}_{\tau,2}$ for the full matrix. Additional evolution matrices may be defined that describe the rotation of the atomic variables of the samples and the effect of the homodyne detection[5]. The update of the system then proceeds as outlined in Sec. II. The theory incorporates the interaction between the atoms and the optical field, atomic decay, and the measurement induced transformation of the atomic state. The reduction of the full quantum state description to a simple Gaussian state fully represented by a set of mean values and a covariance matrix makes the system straightforward to deal with numerically, and analytical results can be obtained in several important cases.

While the general problem of a measure for the entanglement between two mixed states remains unsolved, the entanglement between the atomic ensembles obtained by the continuous probing may be quantified by the Gaussian entanglement of formation[35] or the logarithmic negativity[36]. The Gaussian description may also be used to identify the optimal performance of the entanglement scheme in the presence of atomic decay[5].

A. Entanglement and vector magnetometry

The possibility to entangle more atomic gasses was also considered in magnetometry[28] in connection with the problem of measuring two or three components of the B field with entangled gasses. In the case of two components, say B_y and B_z, the atomic sample is split in two and one gas is polarized along x and the other along $-x$. These polarizations assure that the two observables $(\hat{J}_{y_1} + \hat{J}_{y_2})$ and $(\hat{J}_{z_1} + \hat{J}_{z_2})$, and equivalently $\hat{x}_{\mathrm{at}_1} - \hat{x}_{\mathrm{at}_2}$ and $\hat{p}_{\mathrm{at}_1} + \hat{p}_{\mathrm{at}_2}$ commute. (Note that a different sign convention for the $\hat{x}_{\mathrm{at}_2}$ variable was applied in our previous work[28].) The interaction between the magnetic fields and the two samples is described by the following effective Hamiltonian

$$\hat{H}^m_{\mathrm{int},\tau} = \mu_\tau \hat{B}_y(\hat{x}_{\mathrm{at}_1} - \hat{x}_{\mathrm{at}_2}) + \mu_\tau \hat{B}_z(\hat{p}_{\mathrm{at}_1} + \hat{p}_{\mathrm{at}_2}). \tag{32}$$

This interaction causes changes in the atomic observables $(\hat{p}_{\mathrm{at}_1} - \hat{p}_{\mathrm{at}_2})$ and $(\hat{x}_{\mathrm{at}_1} + \hat{x}_{\mathrm{at}_2})$ proportional with B_y and B_z, respectively. To probe these changes we introduce the effective light-atom interaction

$$\hat{H}^l_{\mathrm{int},\tau} = \kappa_\tau(\hat{p}_{\mathrm{at}_1} - \hat{p}_{\mathrm{at}_2})\hat{p}_{\mathrm{ph}_1} + \kappa_\tau(\hat{x}_{\mathrm{at}_1} + \hat{x}_{\mathrm{at}_2})\hat{x}_{\mathrm{ph}_2}, \tag{33}$$

where the appropriate relative sign between the atomic variables of the two gasses can be implemented by adjusting the sign on κ_τ after the probe beams have passed through the first gas[28]. The gasses are probed by the simultaneous action of the Hamiltonian from Eqs. (32)-(33), $\hat{H}_{\mathrm{int},\tau} = \hat{H}^m_{\mathrm{int},\tau} + \hat{H}^l_{\mathrm{int},\tau}$. The vector of quantum variables is $\hat{\mathbf{y}} = (\hat{B}_z, \hat{B}_y, \hat{x}_{\mathrm{at}_1}, \hat{p}_{\mathrm{at}_1}, \hat{x}_{\mathrm{at}_2}, \hat{p}_{\mathrm{at}_2}, \hat{x}_{\mathrm{ph}_1}, \hat{p}_{\mathrm{ph}_1}, \hat{x}_{\mathrm{ph}_2}, \hat{p}_{\mathrm{ph}_2})^T$. With this state vector and the above Hamiltonian, the formalism of Sec. II can be directly applied and the final uncertainty of the B fields can indeed be lowered compared to the case with individual probe beams by letting the probe beams pass through both gasses and thereby entangling the two. An extension to full three dimensional vector magnetometry using three probing beams and six atomic samples can also be shown to have superior resolution in comparions with measurements on separable systems[28].

VII. EXTENSIONS OF THE THEORY

In this section, we outline some topics which are subject to studies within the Gaussian description at the time of writing, and we discuss how to go beyond the Gaussian approximation.

A. Non spin–1/2 systems

The theory presented in Sec. III explicitly used the representation of the collective angular momentum variable in terms of Pauli spin matrices $\hat{\boldsymbol{J}} = \frac{\hbar}{2}\sum_j \hat{\boldsymbol{\sigma}}$. This representation was crucial for the reduction of the Hamiltonian of Eq. (19) to the form of Eq. (21) which is expressed directly in terms of the canonical $(\hat{x}, \hat{p})$ Gaussian variables. For the more general problem of excited ($|F'M'\rangle$) and ground ($|FM\rangle$) states with $F > 1/2$ and $F' > 1/2$, it is still

possible to obtain an approximate solution within the Gaussian description. This more complicated problem is of both fundamental interest and of practical importance since ongoing experiments based on the Faraday-rotation scheme are carried out with such real multilevel atoms[37, 38].

Equation (17) represents the interaction with atoms with arbitrary level structure, and to deal with such atomic samples within a Gaussian description, we suggest to introduce a second quantized formalism for the atoms in which bosonic atomic field operators $\hat{\Psi}_M^\dagger$, $\hat{\Psi}_{M'}$ create and destroy atoms with the given magnetic quantum number. Note that the bosonic character merely reflects the symmetry under permutations of the atoms: The theory works for both fermionic and bosonic atoms. We can then write the collective atomic operators in the Hamiltonian in terms of the atomic field operators, $\sum_j |FM\rangle_j\langle FM'| = \hat{\Psi}_M^\dagger \hat{\Psi}_{M'}$ to obtain

$$\hat{H}_{\mathrm{int},\tau} = \sum_{M=-F}^{F} [\Big(c_{+,M}(\Delta)\hat{a}_+^\dagger \hat{a}_+ + c_{-,M}(\Delta)\hat{a}_-^\dagger \hat{a}_-\Big)\hat{\Psi}_M^\dagger \hat{\Psi}_M \\ + b_M(\Delta)\left(\hat{a}_-^\dagger \hat{a}_+ \hat{\Psi}_{M+1}^\dagger \hat{\Psi}_{M-1} + \hat{a}_+^\dagger \hat{a}_- \hat{\Psi}_{M-1}^\dagger \hat{\Psi}_{M+1}\right)]. \tag{34}$$

At this point we make a mean field approximation, and we expand the field operators for the light fields

$$\hat{a}_\pm \to \alpha_\pm + \delta\hat{a}_\pm, \tag{35}$$

and the atom fields

$$\hat{\Psi}_M \to \Phi_M + \delta\hat{\Psi}_M, \tag{36}$$

with c-numbers $\alpha_\pm$ and Φ_M, and "small" operators $\delta\hat{a}_\pm$ and $\delta\hat{\Psi}_M$. We insert Eqs. (35)-(36) into Eq. (34) and expand to second order in the operator terms. This procedure leads to a rather lengthy expression which is conveniently split into terms which are of zeroth, first and second order in the quantum fields. The classical fields are explicitly time-dependent and their dynamics is given by replacing all operators by their c-number parts in Heisenberg's equations of motion. Since we neglect operator terms above second order, the quantum part of the Hamiltonian is at most bilinear (with classical time-dependent coefficients). The linear terms cause mean drifts of the mean value of the operator components, which we can absorb in the c-number components. For the new quantum operators, we may then maintain $\langle\delta\hat{a}_\pm\rangle = 0$ and $\langle\delta\hat{\Psi}_M\rangle = 0$. In the resulting bilinear Hamiltonian involving the operator terms, we now make the transition to the Gaussian state description by forming a vector of variables $\hat{\boldsymbol{y}} = (\hat{x}_{-F}, \hat{p}_{-F}, \ldots, \hat{x}_F, \hat{p}_F, \hat{x}_+, \hat{p}_+, \hat{x}_-, \hat{p}_-)^T$, with $x_\pm = \sqrt{\frac{\hbar}{2}}(\delta\hat{a}_\pm + \delta\hat{a}_\pm^\dagger)$, $p_\pm = -i\sqrt{\frac{\hbar}{2}}(\delta\hat{a}_\pm - \delta\hat{a}_\pm^\dagger)$, $x_M = \sqrt{\frac{\hbar}{2}}(\delta\hat{\Psi}_M + \delta\hat{\Psi}_M^\dagger)$, $x_M = -i\sqrt{\frac{\hbar}{2}}(\delta\hat{\Psi}_M - \delta\hat{\Psi}_M^\dagger)$, and adopting the formalism of Sec. II.

We note that the expansion of noise terms around classical mean values has been used as a standard tool in quantum optics, e.g., to deal with the optical Kerr-effect, and if only unitary dynamics and losses are considered, the present approach does not offer any new insights. It is important to remember, however, that we are also able to treat the dynamics conditioned on measurements on the system. Work is in progress, and we will report on results of this approach to the multilevel problem elsewhere.

B. Quantum correlated light beams

So far, we have treated the case of a coherent, monochromatic beam of light incident on the atomic samples. For high precision probing, atomic spin squeezing and entanglement, it has been proposed to use squeezed beams of light and twin beams, and this is indeed also possible within the Gaussian formalism. To model in a simple manner the coupling to squeezed light beams, one may simply alter the covariance matrix elements for the field operators in each beam segment prior to the interaction with the atomic sample, so that rather than the 2×2 identity matrix with equal variances of the two field quadratures in Eq.(11), we assume the form

$$\mathbf{B} = \mathrm{diag}(1/r, r), \tag{37}$$

where r is the squeezing parameter. Carrying out the calculations as described in the previous sections, we observe[7, 28], that the magnetometer resolution is improved by this parameter.

As pointed out in our analysis[7], however, a squeezed beam of light carries correlations between the the field operators evaluated at different times. This implies, that if one observes the beam for only a very short time, one will not be able to detect the squeezing. It is well-known from the standard quantum optical analysis of the optical parametric oscillator (OPO) (in the Heisenberg picture in the frequency domain) that there is a certain frequency

band-width, Γ, of squeezing associated with the field decay rate of the cavity. Only the accumulated signal over sufficiently long times ($t \gtrsim 1/\Gamma$) will be able to extract the frequency components for which the field is squeezed. One might think, that one should therefore consider a longer string of light segments with appropriate covariance matrix elements, and carry out the update on all segments (and the atoms) every time a single optical segment is detected. In fact, there is an easier approach. The squeezed beam is produced by continuous leakage of the field inside the OPO cavity, and the temporal correlations are due to this joint source of the radiation. The most economical way to describe the interaction of atoms with a squeezed beam is therefore to incorporate the single-mode field inside the cavity in the Gaussian state formalism, and to consider again only one optical beam segment at a time, from its creation out of the cavity, interaction with the atoms, and final detection, and hence causing an update of the joint atom and cavity covariance matrix.

We have implemented such a model[39] and verified that it reproduces the known noise properties for the signal integrated over both short and long times. In addition, we have applied the model to magnetometry, and observed that for segments shorter than the inverse band-width of squeezed light, the resolution is not improved with the use of squeezed light, but after many segments and a long total interaction time, the results asymptotically approach the factor $1/r$ improvement of the simple model as one might have expected [39].

In closing this section, we note that the Gaussian description is not restricted to the examples and extensions discussed above. Extra physical systems can be included straightforwardly by adding appropriate rows and columns to the covariance matrix. In this way, one may, e.g., describe the effects of imperfect detectors and filters, and loss in optical fibres.

C. Beyond the Gaussian approximation

Looking back on the development of the theory in this work, we note that the interaction and the dissipation can be treated exactly without recourse to a Gaussian ansatz for the quantum state. In fact, Eq.(5) is a general update formula for the Wigner function under an arbitrary measurement, but in the general case this expression may be difficult to evaluate, and in particular to use as input in the next step of the continuous probing on the system. In a recent experiment[40], a Gaussian squeezed state was mixed with the vacuum field at a beam splitter to produce an entangled two-mode field state part of which was sent to an avalanche photodiode and part of which was monitored by homodyne detection. The state of the second component conditioned on a photodiode counting event is a non-Gaussian state, as verified by a double-peaked homodyne detection signal. The process was modeled[41] by assuming that prior to the registration of a single photon, the state of the field is described by the Gaussian Wigner function $\mathcal{W}_{\rm Gauss}(\gamma,\delta)$ with γ and δ denoting pairs of real variables of the two modes. The detection of a single photon, $|1\rangle$, corresponds to application of the Wigner function $\mathcal{W}_{meas}(\delta) = \mathcal{W}_{|1\rangle\langle 1|}(\delta) = \frac{2}{\pi}\exp(-2|\delta|^2)(-1+4|\delta|^2)$ in Eq. (5),and the Wigner function for the second beam conditioned on this state, $\mathcal{W}_c(\gamma) = \int \mathcal{W}(\gamma,\delta)\,\mathcal{W}_{|1\rangle\langle 1|}(\delta)\,d^2\delta$ is readily shown not to be a Gaussian. Another example where one 'jumps' out of the Gaussian states is in a recent proposal[42] where a setup of two beam splitters with carefully chosen properties and photodetectors allows one to produce with high fidelity a single-photon state from Gaussian squeezed vacuum input beams. We believe that a combination of the theory of Gaussian state updates and inclusion via Eq. (5) of one or a few non-Gaussian preserving measurements may be a useful approach to these problems.

As it is necessary to leave the Gaussian states to perform some quantum information tasks such as distillation of entangled states[2, 4, 43], it is in general important to have tools to handle the interface between Gaussian and non-Gaussian states.

VIII. CONCLUSIONS AND OUTLOOK

We have presented a general formalism to treat the dynamics of $2n$ canonical variables within a Gaussian description. For a probing light beam, the concept of "segment quantization" where the beam is quantized in small fragments of duration τ and length $L = c\tau$ allows an efficient description of not only the evolution of the system subject to the Hamiltonian, but also to the measurement process through simple update formulae for the mean value vector and the covariance matrix, which fully characterize the Gaussian state. Of particular current interest in the field of quantum information and quantum communication[10, 30, 32], and in precision magnetometry[24], is the off-resonant probing of ensembles of atoms leading to a dispersive Faraday effect. This interaction is bilinear in the effective canonical variables of the system, and a description within the Gaussian framework of, e.g. spin squeezing[20], magnetometry[7, 28], and entanglement[5] is straightforward.

In Sec. VII, we have outlined some possible extensions to the Gaussian description. In the future it will be interesting to develop further theory for continuous variable system which leave the Gaussian description, either because of the

interactions involved, because of the measurement schemes, or because of coupling of a small discrete system to the collective continuous degrees of freedom (examples: single photons can be stored and emitted on demand by macroscopic atomic samples, trapped ions can be entangled by continuous probing with classical laser fields). Such approaches hold the potential to form Schrödinger Cat states, which may have favorable properties in high precision detection, and they may be used to implement distillation and purification protocols from quantum information theory, which are known not to work for Gaussian states and operations. The theoretical task is to identify processes that can be implemented experimentally and which break the Gaussian character, and to establish a theoretical description of the resulting states, which will invariably be much more complicated to deal with than the Gaussian states.

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